Inventing Center

Unit

2

Mathematical Concepts

- Statistics measure characteristics of a distribution, such as its center and its variability.
- Different measures of center (e.g., mean, median and mode) emphasize different perspectives about the central tendency of a distribution. The mean represents a "fair share" perspective, the median the 50^{th} percentile point, and the mode the most frequent value.
- An algorithm is a computational process with a finite number of steps that generates a reliable outcome.

Unit Overview

In Unit 2, a simple question is posed: What is the "best guess" of the real length of (name-of-person)'s arm span? The activities in this unit involve students in the design of a measure of center. Students participate in the important mathematical practice of inventing an algorithm.

Day 1: Measuring Center
Using displays developed in the previous unit, students invent a method for finding the "best guess" of the actual length of the person's arm span. Methods should be clearly communicated and if followed, everyone should obtain the same value.

Day 2: Comparing Methods: Measure Review
Next, students who did not author the method try to use it. Different methods are compared and contrasted, with an eye towards discerning which aspects of the data the author noticed. During this activity, you may find it useful to introduce traditional measures or traditional names for student-invented measures (see Mathematical Background).

Day 3: Exploring Traditional Measures of Center
Students explore the median and the mean. After inventing statistics, students are more likely to understand the rationale for multiple measures of center.

Days 4 and 5: Formative Assessment
Students respond to a brief quiz. Student responses that represent different ways of thinking according to the Conceptions of Statistics construct map are deliberately compared and contrasted.

Contents

Materials & Preparation

Read

- ☐ **Unit 2**
  Start by reading the unit to learn the content and become familiar with the activities.
- ☐ **Mathematical Background**
  Reread the mathematical background carefully to help you think about the important mathematical characteristics of these measures.
- ☐ **Sample student thinking**
  Reread Student Thinking, pp. 7-9, to anticipate the kinds of ideas that typically guide students' inventions of measures of center.
- ☐ **Conceptions of Statistics construct map**
  Read the construct map and/or visit the website (Modelingdata.org) to view a progression of student thinking about statistics, beginning with qualitative looking at data and in this unit, culminating with understanding statistics as measures of distribution. Pay particular attention to levels 1-3 on the construct map.
- ☐ **Read Discussion Guide and Measures of Center Planner on Modelingdata.org** to plan a whole-class conversation where students compare their inventions and rationales for measures of center.

Gather

For the class

- ☐ Student displays (from Unit 1)
- ☐ Plain white paper (or computers) for writing directions

Mathematical Background

We briefly explore the mathematics of the statistics describing center—more specifically, some of the measures of the central tendency of a distribution of values. Each measure (statistic) has strengths and weaknesses. There is no universal way to estimate central tendency.

What is a mode?

The mode is the most frequent value. Its logic is that if a value is observed more frequently, it is likely a good estimate of the central tendency in a set of data. Many distributions have more than one mode, so it is often not clear which value to choose. However, the mode is often a sensible choice. For example, for a shoe store, the most frequent foot size might be of commercial significance.

What is a mean?

The mean is a center-point that balances the distribution in the following way: It is the point that minimizes the sum of squared differences between each measurement (observation) and itself. For example, suppose that the measurements of the length of something were 2, 4, and 6. The mean is computed as the sum of the measurements (12) divided by the number of measurers (3). It is a ratio, giving a sense of the value "per" measurer. It is also a "fair share" of the sum apportioned equally among all cases. Here it is 4, suggesting that "on average," each measurer obtained a measurement of 4 units.

If we consider the difference between each measurement and the mean, we find the following:

$2 - 4 = -2$
$4 - 4 = 0$
$6 - 4 = 2$

Notice that the sum of the differences is zero. This follows from the definition of the mean. The sum of the squared differences $(-2)^2 + 0^2 + (2)^2 = 8$. No other measure will have as small a sum of squared differences. It minimizes the total area (each squared difference can be thought of as defining a square of that length) of these differences. Its primary virtue is that it is used by many other statistical techniques that employ a similar, least squares approach.

The mean is very susceptible to extreme scores. For example, replace 6 with 12 and notice that the mean now stretches toward the highest value. Hence, mean values can be misleading. For example, reports of average income that use the mean are often overly optimistic, because they include a few very wealthy individuals.

What is a median?

The median is a center-point that balances the number of cases (observations or measurements) so that half are above the median and half are below. For an odd number of cases, the median is found by ordering the data and selecting the (n +1)/2 case. When the number of cases is even, the median is defined as the midpoint between the 2 cases that define the lower and upper half of the distribution, respectively.

The median is a center-point that minimizes the sum of differences between each measurement (observation) and itself. Hence, it minimizes the total distance. Consider measurement values of 2, 4, and 6 units. The median value is 4, and if we consider the absolute values (the magnitudes) of the differences, then the total difference is 4 (2 + 0 + 2). The median is less susceptible to extreme values compared to the mean, a quality that is often called "robust."

Measuring Center

In this activity, students recognize a need for measuring the center of a distribution. Often, students will quickly recognize the center as worth measuring, but miss the need for a method that is replicable. For example, students might say that the "number in the middle of the clump" is the "best guess." This is an opportunity to develop students' understanding of an algorithm, a clear method others can follow. Students invent and write algorithms that are sufficiently explicit so that other students can follow them. In doing so, they have the chance to learn about two mathematical concepts simultaneously: (1) the measurement of center of a distribution, and (2) developing a measurement method others can follow.

Measuring Center
Comparing Methods
Traditional Measures of Center
Formative Assessment

Whole Group

1. **Introduce the activity: Inventing a method for measuring center.**

   a. Remind students that different measures were recorded in Unit 1. Ask them what they think the "best guess" of the true length of (name-of-person)'s arm span is.

   b. Ask questions like the ones below to draw students' attention to the center of the distribution.

   Q: Do we really need all of these data? Can't we just take one person's measurement as the "best guess" of arm span?
   Q: Would it be ok to just use the largest measurement as the "best guess?" Why? Why not?
   Q: What kinds of things should we think about when we consider how to find the "best guess?"
   Q: How do our displays help us think about the "best guess?"

   Note: The displays students made or others that they considered should be visible, because thinking about the shape of the data helps students reason about how to measure it. However, students often notice the center but define a measure that is not replicable. For example, students might say "the one in the clump" or "the most reasonable one." Use student inventions like these to highlight the need for a clear method—directions (an algorithm) that will lead everyone to the same value.

Construct: CoS1(a)
This discussion often reveals what students notice about the visual quality of the data.

c. Tell students they will work in pairs to invent a method for determining the "best guess" of (name-of-person)'s arm span. Take a moment to make sure students understand that their invented method should:
   - give a number for the "best guess"
   - give the same answer no matter who uses it

   Note: A measure of some characteristic of data (for example, the center) is called a statistic. This might be a good time to introduce this word and begin to use it in class.

Pairs

2. Give students time to invent a "best guess" method.

Put students into pairs to invent an algorithm for "best guess" of center. Provide each pair with a piece of paper to write the directions for their method. These directions will be shared with the class later, so make sure they know to write clearly.

Construct: CoS2(a), CoS3(a), and CoS3(b)
This activity engages students in inventing a measure of center. In particular, this activity can focus on moving students from CoS3(a) to CoS3(b).

3. Listen and watch for methods to highlight in the whole group discussion.

a. Walk around and pay close attention to the types of methods being invented. Use this time to begin thinking about methods that will be productive to share with the class during the next activity.

   Note: Inventing an algorithm is likely something students will not be familiar with, despite their long history of using algorithms in arithmetic. Very often students will not understand at first what they should be doing, or they have difficulty coming up with ideas. If you notice many groups struggling, take the time to remind the whole class what a "best guess" algorithm should do. It is often productive to let students invent *for about 10 minutes* and then to re-convene the class to share some approaches and challenges. It is usually helpful to direct students to look at one or more displays to help them consider the shape of the data. Then, let students return to invention.

b. **Ask questions like the following to help students examine their methods to see if they follow the criteria laid out earlier.**

Q: Can someone else follow your directions and come up with the same “best guess”?

Note. Students often think their directions are unproblematic until later in this unit when others literally try to follow their directions, but it is a good practice to make this criterion of clarity explicit by posing this question.

Q: Is your method a good one? What makes you think so?
Q: Do you think it would work with other data?

Note: Do not be afraid to give students plenty of time to invent their methods. Valuable learning about the center, as a measurable characteristic of a distribution, and about the challenges of developing an algorithm can happen during this time.

Students’ Ways of Thinking: Measuring center

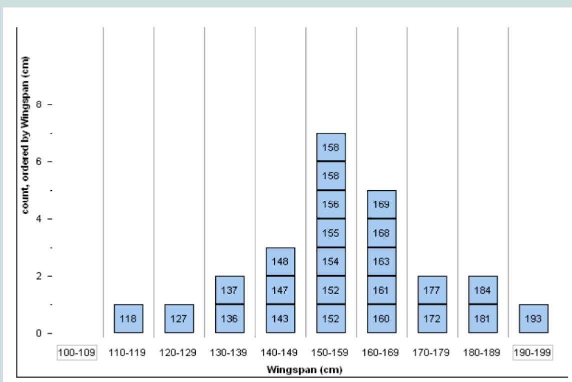
Students’ thinking about how to measure the center or central tendency in measurement data generally falls into four different types of reasoning: (a) convention, (b) repeated values, (c) center clump, and (d) mid-range. Each form of thinking provides an opportunity to help students develop more powerful mathematical ideas.

Convention. Some students have been taught that if they have a batch of data, they should calculate a statistic, usually the mean. They are typically unaware of any of the properties of the mean, so it is a good practice to alter instruction to make some of the qualities of the mean more visible. For example, students can investigate how extreme values affect the mean. Students can also explore how means represent a “per case” point-of-view, if they have a good grasp of ratio. It is important to ground the computation in making sense of the data. Ask students to look at the shape of the data and justify their choice of statistic. See Exploring Traditional Measures of Center, p 13 for some ideas about stretching student understanding of the mean.

Repeated Values. Many students reason that if two or more people agree about a measured value, then that value is more likely the "right" one. Hence, they suggest that the mode is the “best guess” of the true measure. However, some students look for repeated values even when the data are multimodal. For example, the method proposed by one team of fifth-grade students for a batch of data with two modes is displayed below. Notice that it asks users to choose the most "reasonable" value. Invented methods such as this one provide an opportunity to emphasize that algorithms must be clear enough so that anyone who follows them obtains the same result.

1. Sort into groups of ten
2. Find the 150s group
3. Order/sort from least to greatest in 150s group
4. Identify doubles (two 152s and two 158s)
5. 152 cm. is the "best guess" because it is reasonable

Center Clump. Many students who do not have a prior orientation to the mean will invent approximations to the median. Their reasoning is guided by the appearance of a center clump in the data when the data are grouped and ordered.



Most solutions involve finding the middle value of this center bin, a workable solution for measurement data. However, some students independently invent the median, guided by a sense of middle as splitting the data into two parts of equal count, a 50%-50% split. For example, a pair of fifth-grade students came up with the following method:

1. Data out (the data were on cards, one value per card)
2. Grouping second digit (tens) and order from least to greatest in groups (intervals of 10)
3. Put from least to greatest of a group (order values within each group)
4. Count all cards, and then divide the total by 2
5. Count half of the total from least to greatest—first number
6. Count half of the total from greatest to least—second number
7. Find the middle value between the first and second numbers (the sample size was an even number)

Measuring Center
Comparing Methods
Traditional Measures of Center
Formative Assessment

Using the median to spark a shift from case to aggregate. Although conventionally the median makes good sense, many students find it problematic. For example, in one fifth-grade classroom, some objected when the median value was not the same as an actual measurement. How could a non-existent value represent the data? They were persuaded by the inventors' appeal to the measurement process: The median represented a value that might easily have been someone's actual measurement. It was a *possible* measurement. This form of student reasoning signaled a shift away from considering only cases toward considering the aggregate.

A note about the median. The role of ordering values to obtain the median is often not well understood. One potential way of helping students see the value of ordering and of counting ordered values is to re-arrange the values so that the "middle" values of the list represent values found at the tails of the distribution. Then ask students if they believe that it is sensible to represent the actual length of the arm-span (or whatever attribute is being measured) by an extreme value. For example, if the values were 5, 15, 16, 17, 24, the list might be re-arranged to be 15, 16, 5, 24, 17. In this case, 5 splits the data into 2 groups of equal number, but 5 is a poor representation of the center. See Exploring Traditional Measures of Center, p. 13, for an activity about the median.

For an odd number of cases, the median is one of the case-values, and occasionally students find the dual role as a case and as an indicator of "best guess" confusing. Students sometimes suggest that only an odd number of cases can have a median, because only then can the number of cases be split in half. It is important to emphasize that a median splits the cases into two 50% regions. As we mentioned, some students find it helpful to think about the median for an even number of cases as a "possible" value – a value consistent with the center clump of the data.

Approaches similar to the median. Some students order data and find the center of the ordered data but do not consider relative frequency. To them, relative frequency is not as important as the actual values of the cases, so they do not count multiple instances of the same case value. We have found it helpful to ask these students to consider again the process of measurement: Why would we discard some measures? What if not using all the data moves the "best guess" out of the center clump?

Mid-Range. Some students have a distance-based sense of center. They find the difference between the least and greatest value, and then find the mid-point of that distance. This is called the mid-range. (Adding lowest and highest values and dividing by 2 gives the same result.) For example, one student's method was:

1. Subtract from the greatest to least
2. Find ½ of the difference
3. Add ½ of the difference to the lowest number

This idea works well in many circumstances, but it is very susceptible to sample-to-sample variation. For example, one very extreme measurement might shift the value outside the center clump. A teacher might want to ask students to try out a sample with one "wild measurer" and the rest not. This introduces students to the concept of robust statistics—statistics that don't vary a lot from sample to sample, when the variation is just due to chance.

Comparing Methods: Measure Review

Measuring Center
Comparing Methods
Traditional Measures of Center
Formative Assessment

The purpose of this activity is to support students to better understand student-invented, as well as conventional methods, as different ways to measure the center of a distribution. At the beginning of the activity, each group shares their "best guess" algorithm with another group. After trying out someone else's method, groups report back to the class. The students should answer three questions:

Q: Is the method clear (i.e., does the method generate a reliable outcome no matter who follows the method)?
Q: Which parts or characteristics of the distribution does the method attend to?
Q: Does the method result in a good estimate of the real value of the arm span?

Whole Group

1. **Introduce the activity: comparing methods.**

   Ask each pair of students to pass their algorithm for calculating a "best guess" along to another student group.

Construct: CoS2(a), CoS3(a), CoS3(b), and CoS3(c)
This activity engages students in thinking about generalizing the use of statistics.

Small Group

2. **Let students try out the algorithms other students have invented.**

3. **Listen and watch for algorithms to highlight in the whole-group discussion.**

   Circulate around the room and push students to think carefully about the other group's method by asking questions like:

   Q: What is the main idea behind the other group's method?
   Q: Is the method clear? Is it repeatable?
   Q: Does their method make sense? What part of the distribution does the method focus on?
   Q: In what way are these methods the same? In what way are they different?
   Q: Do you think their method would work if the data were different?

Note: The important mathematical concepts are different for each method. For example, the mode only focuses on agreement (repeated values) while the median focuses on center and ignores agreement. Watch for student-invented algorithms that are both similar to and different from traditional measures of center --mean, median and mode. These are good targets for the whole group comparison.

In addition, anticipate the possibility that students will not be able to think of scenarios where the methods do not give good estimates. In this case, many teachers provide an example that highlights the problem. For example, you might ask students to find the median and mean of a distribution with one large outlier. If students have measured the same attribute twice, once with a crude tool and once with a better tool (e.g., one that requires fewer movements to find the measure), then the invented methods could be applied to both sets of data. Usually, the centers will be very close, because the true value of the measure of the attribute has not changed. Or, methods could be tried out on other measurement data or on some of the production data in Unit 4.

Whole Group

4. Students report back to the class about results of the comparison.

a. Ask one pair of students to explain their method or that of another pair's method to the class.

   Note: You should select pairs deliberately. Make sure students have opportunities to compare different mathematical approaches to measuring center. Often, as few as two carefully selected measures of center can lead to productive discussions.

b. Use questions like the following to get students thinking and talking about the mathematical elements of the method after it is shared.

   Q: Is this a good method for determining the "best guess" of the actual length of (name-of-student)'s arm span?
   Q: What parts of the data does this method use?
   Q: Can anyone think of a situation where this method might not give us a very good "best guess?"

c. Select additional groups to report back. Use questions like the ones below to get students talking about the differences in the method. Work to make sure that students notice and talk about the mathematical differences in the methods and how these differences affect the measure in different scenarios, such as outliers.

Q: How is this method like this other one? How is it different? What do you think the author intends?
Q: Some of us proposed finding the mean (if anyone did). What will happen to the mean if we include a measurement that's out here (an outlier)? What about the median? The mode?
Q: Some of us proposed finding the median (if anyone did). (Name-of-student) says the median is the middle number. Someone says this means that the median of 2, 7, 10, 5, 3 is 10. What do you think?
Q: Is the most common measurement always the "best guess?"
Q: Which of these methods would work with other data too? Why do you think so?

Note: Teachers should be alert to thinking that seems to be guided by the "center clump," perhaps asking students why they think the largest clump is at the center (most data sets collected in this manner will have a bell-shaped distribution). Students should be encouraged to consider traditional measures of central tendency (mean, median, mode) if none are proposed. See Exploring Traditional Measures of Center for additional activities related to median and mean.

Exploring Traditional Measures of Center

It is highly likely that students have invented measures that are similar to or the same as traditional measures of center. In this activity, you can have your students explore how ordering and counting can be used to find the median and how "fair sharing" can be used to find the mean.

Exploring the Median

Distribute the *Exploring Measures of Center: Median* worksheet.

Construct: CoS2(a)

Introduce the activity by saying:

> Almost all of us thought the best estimate of the length of (name of person)'s arm-span was somewhere in the center of all of the measurements. We developed a number of strategies to come up with our "best guess." One strategy a few of us proposed was to find what people in math call the median—the value that splits the measurements into two equal parts. Please find the median of the sets of measurements on the worksheet. You and your partner must agree about the value for the median.

Thought-revealing questions:

Q: People say that the median is a middle number. If that is true, why is 43 not the median for problem 1?

Q: How is finding the value of the median different for an even and odd number of values?

Exploring the Mean—the Fair Sharing Context

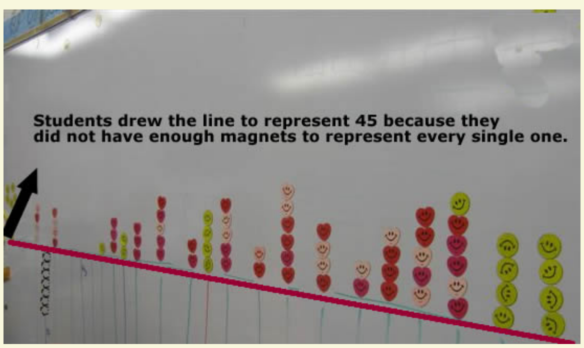
Each member of the class receives a bag of M&M's. Students count the number in each bag. The question is: What is the target number of M&M's per bag? In one class, the lowest number of M&M's was 38 and the largest count was 51. To estimate the target number of M&M's for each bag, several students proposed a "middle number." This included mode and median as ways of thinking about middle numbers. But their teacher wanted to include a way that none of them had thought about: the mean.

Classroom Talk: Introducing the mean as fair sharing

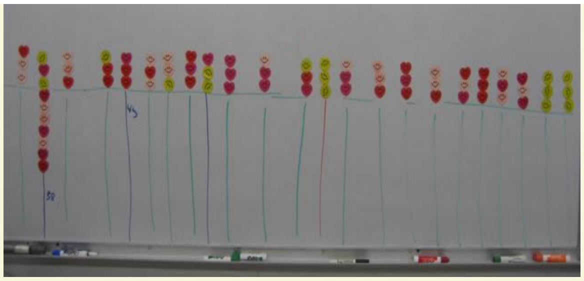
The big idea about the mean the teacher wanted to support was that the values of the batch of data are pooled (the sum) and then distributed fairly to each case represented in the batch of data.

Different numbers of M&Ms are redistributed equally by fair-sharing.

To introduce fair-sharing, the teacher asked if it was okay that an older sister got more M&M's than a younger sister. A student said: "That wouldn't happen in my house. My mom would put them all in one bowl and then divide them back evenly. One for you and one for me. . .so that we would have the same share. That's fair." The class created a display on the whiteboard to represent sharing all the M&M's fairly. All students had bags with 45 or more M&M's, except for one bag that had 38. The class drew a line to represent 45 and then used magnetic stickies above the line to represent additional M&M's, and drew 7 circles below the line to represent the "weird" bag (the one with a count of 38).



Students then redistributed the magnet-stickies so that everyone had the same number (the bag with 38 received the most)—a fair sharing, as displayed below. The resulting value, 48, was the mean. The mean represents a redistribution of the total (the sum) so that each case (bag) receives the same amount—a fair share!



Numbers of M&M's in each bag are redistributed equally, with the result of 48.

Extensions to this activity could include non-whole number fair-shares.

You might have the students pour the M&M's from each bag into a big bowl, and then redistribute them so that everyone receives the same number of M&M's by dealing them one by one. This process of partitioning could then be related to the algorithm for computing the mean as the total/number of contributors.

Extending the interpretation of the mean as a fair share.

Distribute the *Exploring Measures of Center: Mean* worksheet.

Note: The mean of the measurements represents a length obtained by fair-sharing. The mean length can be represented literally by cutting strips of paper to represent each length, gluing all the strips together into one composite length, and then partitioning the composite length into n equal-size partitions. This works best when n is a multiple of 2. Each equal-length partition represents the mean, so that n times the mean length is equal to the total length of the composite strip. Students often find that this activity is revealing. The worksheet has 4 measurements. You may wish to expand this to all, or nearly all, the measurements taken by the class, but please be certain that the number of measurements is a factor of 2.

Formative Assessment

1. **Administer the quiz** (pp 19-22, or download from Modelingdata.org)**.**

   Exercise Ball & Swimming Strokes help students consider alternative methods for measuring center, and the implications of those methods.

2. **Use the scoring guide to score student responses.**

3. **Use Exercise Ball responses to generate a discussion of statistics and distributions.**

   a. **Select student responses to compare and contrast.**
   While scoring, carefully select student responses to compare and contrast during the conversation. Pick responses that show reasoning at different levels of the construct map. These responses should include one from each level on the scoring guide. This item does not have many levels for each part, but different strategies can lead to the same level, so keep an eye out for multiple ways students thought about the items. For example, students might use different strategies for questions 2 and 3 to reason relationally about measures of center. These are important to highlight in the conversation.

   b. **Prepare questions to support and guide student thinking.**
   For example:
   - Some ordered the data to find the median while others did not, does this matter? Why?
   - Why are these three best guess numbers different? Which would you use with this data?
   - What do you know about the mean? If you know the mean and the sample size, what else do you know? Suppose you know the total and the mean, what else do you know?

   c. **Use an Assessment Conversation to help students consider statistics in relation to the distributions they are meant to describe.**
   The responses should be presented to the class by their authors. The teacher should guide the conversation with questions that direct the students to important elements of the statistics. For example, for question 3, one strategy is to use the relation of mean and N to find the total sum and then choose any two values that satisfy that sum. Another, balance strategy is to know that the sum

of the deviations from the mean must sum to zero. Hence, (44-46 = -2), (49-46 = +3), and (47-46 = 1), so that two other scores must be chosen that satisfy a deviation of -2 so that the sum of deviations is 0. For example, (44- 46 = -2 and 46-46 = 0) are one such pair, but so too are (40 – 46 = -6 and 50 – 46 = 4).

4. **Use Swimming Strokes responses to generate a discussion of what displays show and hide.**

   a. **Select student responses to compare and contrast.**
   While scoring, select (at least) two responses for question 1, one in favor of the mid-range and another opposed because it is not in the center clump. Select (at least) two responses to question 2, one of which is the median (if available) and another that focuses on the mode (more common). Select (at least) two responses to question 3, one of which considers how the other measure improves upon the mid-range (e.g., the mid-range only relies on two values, the mid-range misses the center clump but the other measure is in the center clump) and one of which tends to use less relevant criteria, such as "I thought of it."

   b. **Prepare questions to support and guide student thinking.**
   Examples of questions to use during the conversation:
   - What do we want to measure here? (best guess of real number of swimming strokes)
   - Why might the measurements not all be the same?
   - Looking at the display, where do you think the real number of strokes is most likely to be
   - Why might we want to have a method for best guess of the real number of swimming strokes that uses more than two of the measurements?

   c. **Use an Assessment Conversation to help students consider what displays show and hide.**
   When contrasting, highlight the importance of considering not only what displays show, but also what they hide.

Question 1	Highlight how the estimate of best guess is found. Emphasize that its sense of "middle" is that of splitting the total range into two equal parts. Highlight potential drawbacks to a best guess determined by the two most extreme measurements.
Question 2	Highlight how median, mode, or mean use

more of the data to obtain a best guess—and how each appears to be located in the center clump. Consider how each has a different sense of center. The mode says that repeated measurements are more trustworthy. The median says that the middle splits the data into two equal counts. The mean fair-shares all the measurements.

Question 3	Add an outlier to the data and discuss its effects on each estimate of the center—the real number of swimming strokes.

Exploring Measures of Center: Median

Find the median of each of the following sets of measurements. Show your work.

1. 50 47 43 45 44

2. 21 18 19 109 23 27 2

3. 67 65 62 64 61 61

4. 17 24 24 25 26 26 26 27 29 31 17 17

Answer each question:

(a) How is finding the value of the median different for an even and odd number of values?

(b) For an even number of values, the median is often not a value found in the data. So, how can a value not in the data represent the central tendency of the data?

Exploring Measures of Center: Mean

Here are the measurements of the height of a package in cm: 11, 10, 12, 7.

For each measurement, cut an equivalent length paper strip. Join the paper strips together with tape to create a total length. Then split the total length into 4 equal parts (take ½ of the total length by folding, then take half of the remaining length). Measure the split length. Relate what you did to the following: The mean is the sum/n. How is what you did like when you fair-shared the M&M's?

Name: ____________________________________

Grade: ____________________________________

Teacher: ____________________________________

Gender: Male (boy) ________ Female (girl)________

Language you speak at home:__________________

Unit Quiz

Exercise Ball

A group of 7 students measured the circumference of an exercise ball. Here are their measurements in inches:

42, 46, 45, 47, 43, 46, 46

1. Find the median, mode, and mean and enter your answers below. Show your work.

   The median is _____. The mode is _______. The mean is _______.

2. Tom forgot to put his measurement on the list. When the students added Tom's measurement to the list the mean and the median decreased, but the mode stayed the same. Which value is most likely to be Tom's measurement? Circle your choice:

   a. 43
   b. 45
   c. 46
   d. 47

   Explain why you chose this measurement:

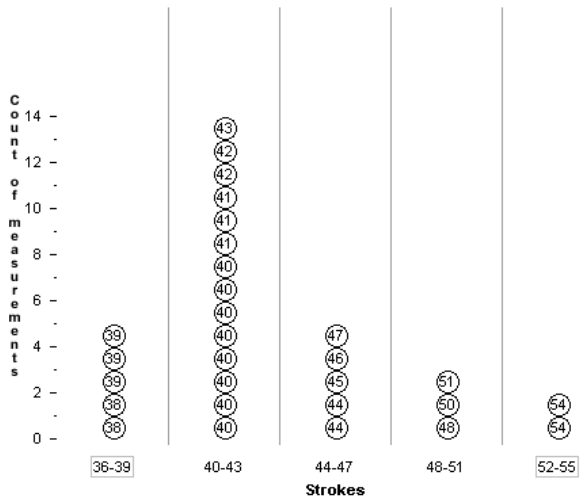
3. Five students in another classroom measured the circumference of the same ball. Their mean was found to be 46 inches. Of the five measurements, three are provided for you below. What could the other two measurements be, so that all five values will have a mean of 46? Show your work.

   44, 49, 47, ____ ____

Unit Quiz

Swimming Strokes

Twenty-nine people watched Mark swim across the pool. Each person counted the number of strokes Mark took to cross the pool. It was hard to count, so they didn't all get the same count. Next, they made a display of their data.



They want to know **how many strokes Mark actually made.** Someone said, "I think we can decide by finding the **mid-range** of the data. Find the lowest number and the highest number. **46** is in the middle of those two numbers, so this is a good estimate of the actual number of strokes that Mark made."

1. Do you agree that **46** is a good estimate of the actual number of strokes? Explain **why** or **why not.**

2. Using the **same data,** describe another way to estimate the actual number of strokes made by Mark.

3. Which method is better to find the **actual number of strokes**—finding the mid-range of the data or using the way you described in question 2? Why?

Unit Quiz Scoring Guide

Exercise Ball

Question 1a: Median Exercise Ball and Conceptions of Statistics (CoS)		
Level	**Performance**	**Example**
CoS(2a)	**Calculate the statistic for central tendency.** Correctly calculates median (*may or may not show work*).	• "median = 46"
CoS(2a-)	**Calculate a statistic, but no evidence of focusing on function.** Fails to order data before calculating median. Finds the middle number in the unordered set (i.e., median=47).	• "median=47"
CoS(2a--) confusion	Confuses median with other statistics such as mean (must show work).	• "median=315/7=45"*
NL(ii)	Incorrectly selects other numbers from the data (i.e., 42, 43, or 45) as the median.	• "median = 42" • "median = 43" • "median = 45" • "median = (45,46) "
NL(i)	**Response is irrelevant, unclear, or a restatement of given information.** Writes numbers that are incorrect and not plausible values.	• "9" • "22" • "7" • "106" • "Cool" • "I don't know."
M	Missing response	

*Mock student responses

Question 1b: Mode Exercise Ball and Conceptions of Statistics (CoS)		
Level	**Performance**	**Example**
CoS(2a)	**Calculate the statistic for central tendency.** Correctly calculates mode (*may or may not show work*).	• "mode = 46"
CoS(2a-) Confusion	**Calculate the statistic for central tendency.** Confuses mode with other central tendency statistics (e.g., mean, median). Must show work.	• Student writes 'mode=45' and procedure of calculating a mean.
NL(ii)	Selects number from the data set provided but this number is not the correct mode.	• "mode = 42" • "mode = 43" • "mode = 45"
NL(i)	**Response is irrelevant, unclear, or a restatement of given information.** Writes numbers that are incorrect and not plausible. Specifically, writes numbers other than the measurements provided	• "mode = 5" • "mode = 30" • "mode = 12" • "Hals"
M	Missing response	

*Mock student responses

Unit Quiz Scoring Guide

Question 1c: Mean **Exercise Ball and Conceptions of Statistics (CoS)**		
Level	**Performance**	**Example**
CoS(2a)	**Calculate the statistic for central tendency.** Uses correct procedure to calculate the mean (*may or may not show work*). Student may make computation errors in the work shown.	• "mean = 45, because 42+ 46+ 45+ 47+ 43+ 46+ 46= 315. 315/7 = 45" • "mean = 45" • "Mean = 25, because 42+ 46+ 45+ 47+ 43+ 46+ 46 = 315. 315/7 = 25."*
CoS(2a-) Confusion	Confuses mean with other central tendency statistics (e.g., mode, median). **Must show work.**	• Writes 46 and work shows procedure of calculating the median or mode
NL(ii)	**Relevant but incorrect response.** Uses incorrect procedure for calculating mean, for instances adds numbers but does not divide *OR* Writes a number within the range of the data set provided ([42, 47]) but this number is not the correct mean	• "mean = 42" • "mean = 43" • "mean = 46" • "mean = 47" • "mean = 315, because 42+ 46+ 45+ 47+ 43+ 46+ 46 = 315" • "315"
NL(i)	**Response is irrelevant, unclear, or a restatement of given information.** Writes numbers other than the measurements provided	• "mean = 5" • "mean = 30" • "mean = 12" • "Goah" • "A lot"
M	Missing response	

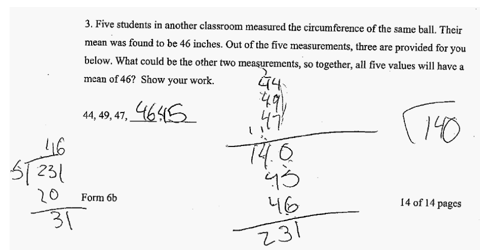
*Mock student responses

Unit Quiz Scoring Guide

Question 2: A forgotten measurement Exercise Ball and Conceptions of Statistics (CoS)		
Level	**Performance**	**Example**
CoS(3d)	**Predict how a statistic is affected by changes in its components or otherwise demonstrate knowledge of relations among components.** Students' values for mean, median, and mode are all within the range of 42-47 (the values given) Student chooses a value that is lower than (not equal to) the values for mean and median that he or she selected in Part 1.	• Student has correctly calculated the values in Q1 and chooses "a" (43) • Student says the mean is 46 and the median is 47 (in Q1), then chooses "b" • Student says the mean is 46 and the median is 45 (in Q1) and chooses "a"
NL(ii)	**Inconsistent choice.** Choose an answer that is **not smaller** than either the mean or median calculated in Q1. *OR* Student calculated values outside the range of 42-47 for Q1. Student may choose any answer	• Student calculates median as 76 and mean as 78 (for Q1), then chooses "a" (43) for Q2. • Student calculated mean and median correctly in Q1 and chooses "d" (47) for Q2 • Student wrote "I don't know" for part 1, and then chooses "a" for Q2.
NL(i)	Irrelevant attempt.	• Student chooses more than one answer or did not choose any option except scribbling.
M	Missing response	

*Mock student responses

Unit Quiz Scoring Guide — Inventing Center Unit 2

Question 3: Find missing values for a mean of 46 Exercise Ball and Conceptions of Statistics (CoS)		
Level	**Performance**	**Example**
CoS(3d)	**Predict how a statistic is affected by changes in its components or otherwise demonstrate knowledge of relations among components.** • Correctly provides any pair of two positive numbers whose sum is 90. For correct answer: may or may not show work. • A student can also be put at level 3D if he or she *shows work and demonstrates understanding of procedure* and approximately correct answers but makes a calculation mistake; the sum for their two measurements will not be 90 but can otherwise be between 85 and 95, excluding 90.	• "42 and 48. 44-46= -2, so I need 48-46=2;49-46 = 3, 47-46= 1, so I need 42-46=-4" • "43, 47" • "43,47. 46*5=230, 230-44-49-47=90, 43+47=90"*  3. Five students in another classroom measured the circumference of the same ball. Their mean was found to be 46 inches. Out of the five measurements, three are provided for you below. What could be the other two measurements, so together, all five values will have a mean of 46? Show your work. 44, 49, 47, 46 45 Form 6b 14 of 14 pages
NL(ii)	Provides a pair that does not sum to 90; either does not show work, or work indicated student did not know the correct procedure of calculating mean. (*Note*: This is different from CoS(2b) in which although student might provide values that did not sum to 90, their work indicated their mistakes were from calculation errors rather than conceptual ones.)	• "13, 12" • "35, 45" • "52, 50" • "46, 46" • "46, 4"
NL(i)	Response is irrelevant, unclear, or a restatement of given information.	• "It is impossible."
M	Missing response	

*Mock student responses

Swimming Strokes

Question 1: Is 46 a good estimate of the actual number of strokes? Swimming Strokes and Conceptions of Statistics (CoS)		
Level	**Performance**	**Example**
CoS(3f)	**Choose statistics by considering qualities of one or more samples.**	• "No, because it only shows the mid-range and not which number of stroke most people got." • "No, I think it is 40 because most people got that number." • "No because it is not in the center clump."
CoS(3f-)	Disagree and reason that 46 is not representative of the sample.	• "No, because there is only one 46."
NL(ii)	Disagree but explanation is not clear or shows incorrect reasoning.	• "I do not believe that because I used a different method." • "If you divide 54 by 2 you get 27, not 46." • "Yes, because it's near the middle." • "Because it is the median of the measure if you only counted the unique ones."
NL(i)	Agree or irrelevant response.	• "I don't know."
M	Missing response.	

*Mock student responses

Question 2: Using the same data, describe another way to estimate actual strokes. *Note:* Sometimes students did not fill out this part, but they gave a method in part A. Swimming Strokes and Conceptions of Statistics (CoS)		
Level	**Performance**	**Example**
CoS(3c)	**Generalize the use of a statistic beyond its original context of application or invention.**	• "Use my method would be real easy. It used the mode. (1) Order the data (2) Make all the bin size 10 (3) Find the highest frequency bin (4) Find the mode of that bin."

		• "If there are 29 numbers and you divide in half you get 14 and 1. [Starting from the least] Count to fourteen is 41. But think of 14 and 1 as 14.5. And there are 2 of them ½ + ½ = 1. 14=15 which is still 41." (Basically he described how to find the median.) • "Choose the mean." • "I think it is 40 because most people got that number." • "Find the median of the set and that's the actual number"
NL(ii)	Propose a method that is not justified, for example, a method that does not use all the values. Treat data as collection of individual numbers.	• "Add all the numbers up and find the one that is in the middle." • "You can count the unique ones and find the median." • "1st put in order 2nd find the mean and mode add them together there you go." • "You could count them."
NL(i)	Attempt item but answer describes no procedure or focusing on making a better display.	• "Make a bar graph." • "Use a tally chart." • "I couldn't find the answer."
M	Missing response.	

*Mock student responses

Question 3: Which way is better to find the actual number of strokes? Swimming Strokes and Conceptions of Statistics (CoS) *Note:* Sometimes students did not fill out this part, but they gave a method in part A.		
Level	**Performance**	**Example**
CoS(3f)	**Choose statistics by considering qualities of one or more samples. Justify the answer with considerations of sample characteristics.**	• "My method is better because it concentrates on the clump that contains most of our counts. The middle of the range isn't even in the central clump."
NL(ii)	Compare two statistics using some criteria that are either not clear, or not justified statistically.	• "My way is more accurate." • "My way is easier." • "It has the most in it." • "My method is better because it will always find a number that exists. But using the method in the problem sometimes you might end up with a number that is not even in the collection!"* • "My method is better because it makes better sense."*
NL(i)	Attempt item but does not compare methods or answer is irrelevant.	• "That is a good question."
M	Missing response.	

*Mock student responses